Distinct Distances with Different Metrics

This document provides an example project of Adam, with too much information. To get a first impression of the project, you might prefer to only read the first couple of pages and quickly skim the rest. You can also write to Adam for more information.

The project is a follow-up to a previous undergraduate research project mentored by Adam. That project was not only published in a non-undergraduate journal, but also won a $\$1,000$ young researcher award of the journal, for researchers under 35 (all participants were under 25!)

1 The distinct distances problem

The *distinct distances problem* was introduced by Erdős in a famous 1946 paper [2]. Consider a set $\mathcal{P}$ of points in the plane. Every pair of points from $\mathcal{P}$ have some distance between them. We are interested in the set $\Delta(\mathcal{P})$ of the distances spanned by all such pairs. For example, let $\mathcal{P}$ be the house-shaped set of five points depicted in Figure 1. Assuming that the side of the square is of length 2, we have $\Delta(\mathcal{P}) = \{2, \sqrt{2}, \sqrt{8}, \sqrt{10}\}$.

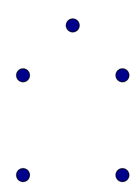

Figure 1: A house-shaped set of five points. If the side of the square is of length 2 then the diagonal of the square is of length $\sqrt{8}$ and each side of the roof is of length $\sqrt{2}$.

Note that every distance appears once in Δ, no matter how many pairs span it. More formally, $\Delta(\mathcal{P})$ is not a multiset. That is why we call $\Delta(\mathcal{P})$ the set of *distinct distances*. Erdős asked for the minimum number of distinct distances that can be determined by a set of n points in the plane. In other words, he asked for

$$D(n) = \min_{|\mathcal{P}|=n} |\Delta(\mathcal{P})|.$$

To clarify, the minimum is over all sets $\mathcal{P}$ of n points in $\mathbb{R}^2$.

For example, n points equally spaced on a line determine $n-1$ distinct distances (see Figure 2). Thus, we have $D(n) \le n-1$.

Let $\mathcal{P}$ be the set of vertices of a regular polygon with n sides, as in Figure 2. It is not difficult to show that $D(\mathcal{P}) = \lceil (n-1)/2 \rceil$.

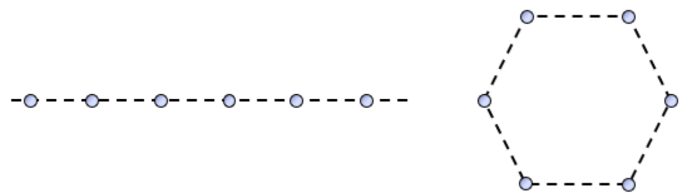

Figure 2: Left: Points equally spaced on a line. Right: Vertices of a regular n-gon.

In Erdős's original paper, he considered the $\sqrt{n} \times \sqrt{n}$ integer lattice

$$\mathcal{P} = \{(a, b) \ : \ 1 \leq a, b \leq \sqrt{n}\}.$$

The number of distances that are determined by this set is an immediate corollary of a result from number theory.

Theorem 1.1. (Landau and Ramanujan) *There exists a positive $c \in \mathbb{R}$ such that the number of positive integers smaller than n that are the sum of two squares is approximately $c \cdot n/\sqrt{\log n}$.*

Every distance in the $\sqrt{n} \times \sqrt{n}$ integer lattice is the square root of a sum of two squares between 0 and n. Thus, Theorem 1.1 implies that the number of distinct distances in this case is about $c \cdot (n/\sqrt{\log n})$.

Theorem 1.2 (Erdős [2]). *There exists $c > 0$ such that $D(n) < c \cdot n/\sqrt{\log n}$ for every n.*

Erdős conjectured that the bound in Theorem 1.2 is tight. That is, that every set of n points in the plane determines at least $c \cdot (n/\sqrt{\log n})$ distinct distances, for some constant $c > 0$. Here is a first example for deriving a lower bound for $D(n)$.

Claim 1.3. $D(n) \geq \sqrt{(n-2)/2}$.

Proof. Let $\mathcal{P}$ be a set of n points and consider two points $v, u \in \mathcal{P}$. Let d_v denote the number of distinct distances between v and $\mathcal{P} \setminus \{v\}$. Note that the points of $\mathcal{P} \setminus \{v\}$ are contained in d_v circles that are centered at v. We denote this set of circles as Γ_v. We define d_u and Γ_u symmetrically. Each of the $n - 2$ points of $\mathcal{P} \setminus \{v, u\}$ is contained in the intersection of a circle from Γ_v and a circle from Γ_u (an example is depicted in Figure 3). Two circles intersect in at most two points, so the number of such intersections is at most $2|\Gamma_v||\Gamma_u| = 2d_v d_u$. This implies that $2d_v d_u \geq n - 2$. When $d_v < \sqrt{(n-2)/2}$ and $d_u < \sqrt{(n-2)/2}$, we have that $2d_v d_u < n - 2$, which is a contradiction. Therefore, $\max\{d_v, d_u\} \geq \sqrt{(n-2)/2}$. □

A large number of works have been dedicated to the distinct distances problem and its many variant. Over the decades, stronger and stronger lower bounds for $D(n)$ have been derived. In 2010, Guth and Katz [4] almost completely proved Erdős's conjecture.

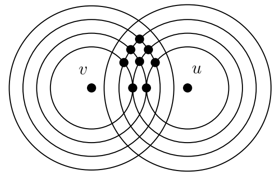

Figure 3: The points of $\mathcal{P} \setminus \{v, u\}$ are contained in the intersections of Γ_v and Γ_u.

Theorem 1.4 (Guth and Katz). *There exists $c > 0$ such that $D(n) > c \cdot (n/\log n)$.*

Note that there is still a gap of $\sqrt{\log n}$ between the bounds of Theorem 1.2 and Theorem 1.4. While this problem is almost completely settled, there are many other wide open distinct distances problems, most of which were also posed by Erdős. For example, Erdős originally asked for the minimum number of distinct distances n points can determine in $\mathbb{R}^d$, while the problem is almost settled only in $\mathbb{R}^2$. The minimum number of distances spanned by n points in $\mathbb{R}^3$ is still wide open.

As another example, Erdős asked if, for every set of n points in $\mathbb{R}^2$, there exists a subset of about $\sqrt{n}$ points where no distance appears more than once. That is, for the subset, all distances are distinct. This conjecture is still wide open. For many other open distinct distances problems, see [6].

2 Distinct distances with other metrics

Why are distinct distances problems so difficult? One philosophical answer is that, while these are combinatorial problems, they are actually about studying the underlying geometry. In particular, small changes in the geometry may completely change the behavior of the problem. This implies that, to solve a problem, one has to use properties of the specific geometry that is considered. In this project, we study how different *metrics* (distance functions) change the behavior of the problem.

In the preceding section, we were using the *Euclidean metric*. That is, we defined the distance between two points (a_x, a_y) and (b_x, b_y) to be

$$\sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}.$$

While this is the most common distance metric, there are many others in mathematics.

In the following, we only consider metrics in the plane. For simplicity, we consider each metric by stating the distance between the points and (a_x, a_y) and (b_x, b_y). For a real number $p \geq 1$, the metric induced by the ℓ_p norm is

$$\left(|a_x - b_x|^p + |a_y - b_y|^p\right)^{1/p}.$$

For brevity, we refer to such a metric as the ℓ_p metric. Note that the ℓ_2 metric is the Euclidean distance.

The ℓ_1 metric is the distance function $|a_x - a_y| + |b_x - b_y|$. This is sometimes called *the Manhattan distance*, since one can think of it as the number of blocks to cross when traveling between two points in Manhattan. If you are on the corner of 3rd Avenue and 25th Street and wish to travel to the corner of 5th Avenue and 34th Street, then you need to travel $|25 - 34| + |3 - 5| = 11$ city blocks.[1]

The *unit circle* of a metric is the set of points that are at a distance of 1 from the origin $(0, 0)$. Under the Euclidean distance, the unit circle is a circle of radius 1 centered at the origin. Under the ℓ_1 metric, it is a square with side length 2 standing on a vertex (see Figure 4).

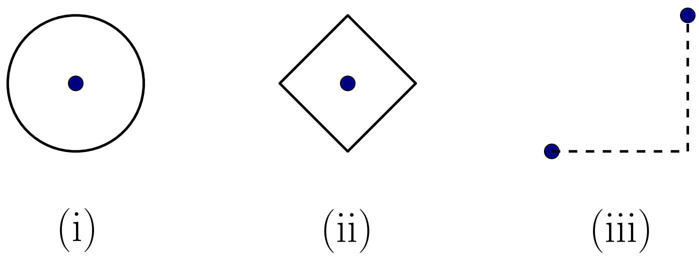

Figure 4: (i) The ℓ_2 unit circle. (ii) The ℓ_1 unit circle. (iii) Distances in ℓ_1 can be imagined as traveling only with vertical and horizontal lines.

The larger p is, the more dominant $\max\{|a_x - b_x|, |a_y - b_y|\}$ is in the ℓ_p metric. Under the ℓ_1 metric, both $|a_x - b_x|$ and $|a_y - b_y|$ have the same influence on the distance. Under the ℓ_2 metric, the larger of the two already matters more when computing the square root. Under the ℓ_3 metric, the larger difference matters even more, and so on. For this reason, the ℓ_∞ metric is defined as $\max\{|a_x - b_x|, |a_y - b_y|\}$.

The ℓ_1 and ℓ_∞ metrics are somewhat degenerate. Mathematicians are usually interested in metrics with a unit circle that is strictly convex.[2] This is the case for all ℓ_p metrics except for ℓ_1 and ℓ_∞. For example, Figure 5 depicts the unit circle of the ℓ_5 metric.

A function $p : \mathbb{R}^2 \to \mathbb{R}$ is a *norm* if it satisfies the following properties:

- If $p(a) = 0$ then $a = (0, 0)$.
- Every $a \in \mathbb{R}^2$ and $k \in \mathbb{R}$ satisfy $p(k \cdot a) = |k| \cdot p(a)$.
- Every $a, b \in \mathbb{R}^2$ satisfy $p(a + b) \le p(a) + p(b)$.

If $p : \mathbb{R}^2 \to \mathbb{R}$ then $\Delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as $\Delta(a, b) = p(a - b)$ is a metric. We say that Δ is the metric *induced* by the norm ℓ_p. Not every metric is induced by a

[1] These are the locations of the CUNY Graduate Center and CUNY's Baruch College. Adam constantly needs to travel between these two locations. When checking this on a map, you'll find out the this is a lie and Adam actually travels more than 11 blocks.

[2] That is, for any point p, q on the unit circle, the line segment between p and q intersects the unit circle only at its endpoints. The interior of this line segment is completely in the interior of the unit circle.

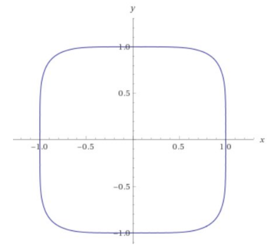

Figure 5: The ℓ_5 unit circle.

norm. However, in our project we are mainly interested in metrics that are induced by norms.

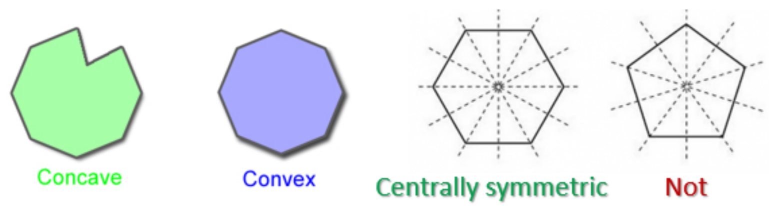

Figure 6: Examples of convex and centrally symmetric shapes.

The unit circle of any metric is a closed curve that is centrally symmetric about the origin. That is, for every pair of opposite directions v and u, when shooting rays from the origin in directions v and u, both rays hit unit circle after traveling the same distance. The unit ball of any metric is convex. That is, for any two points p, q on the unit ball, the line segment between p and q is fully contained in the unit ball. Every convex region whose boundary is centrally symmetric around the origin is the unit ball of a unique norm. The boundary is then the corresponding unit circle.

The following result is from [1].

Theorem 2.1. *For most metrics in $\mathbb{R}^2$ (induced by norms), the minimum number of distinct distances determined by n points is very close to n.*

In this introductory document, we do not rigorously define "most metrics." We only state that the set of metrics that do not satisfy the above has measure zero in the space of all metrics. (When considering only metrics induced by norms.)

While Theorem 2.1 provides information about distinct distances in most metrics, it is not constructive. We know that the ℓ_1, ℓ_2 and ℓ_∞ metrics are exceptions to the statement of Theorem 2.1, all having smaller numbers of distances. We do not yet know of nice metrics that satisfy the statement of Theorem 2.1.

The dissertation of Julia Garibaldi[3] [3] was about distinct distances in non-Euclidean metrics. The dissertation studies various proofs for the case of the Euclidean distance and adapts those to other metrics.

[3]The first graduate student of Terrence Tao.

The project. Garibaldi proved that, for any $1 < p < \infty$, we have that $D(n) > c \cdot n^{4/5}$ for the ℓ_p metric. A previous undergraduate project mentored by Adam [5] improved the bound to $D(n) > c \cdot n^{6/7}$, for integer p. Now, Adam is interested in working on the case of non-integer p. He has concrete ideas for how to potentially address this problem, but prefers not to share those here. We could also explore related problems, such as different types of metrics.

References

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