EXAMPLE REU PROJECT: CONVEX GEOMETRIES VIA CONVEX SHAPES REPRESENTATIONS

A convex geometry is a special case of a closure system. It can be defined through a closure operator, or through a family of its closed sets.

Definition 1. *Given any (finite) set X, a closure system on X is a pair $(X, \mathcal{F})$, where F is a family of subsets of X which satisfies two properties:*

- $X \in \mathcal{F}$
- *If* $Y, Z \in \mathcal{F}$ *then* $Y \cap Z \in \mathcal{F}$.

Definition 2. *A closure system $(X, \mathcal{F})$ is a convex geometry iff the set family $\mathcal{F}$ satisfies the following two properties:*

- $\emptyset \in \mathcal{F}$
- *If* $Y \in \mathcal{F}$ *and* $Y \neq X$ *then* $\exists a \in X \setminus Y$ *s.t.* $Y \cup \{a\} \in \mathcal{F}$.

Very often convex geometries are defined as set systems related to *antimatroids*. $(X, \mathcal{F})$ is a convex geometry iff dual set system $(X, \mathcal{F}^*)$ is an antimatroid.
Here $\mathcal{F}^* = \{X \setminus Y : Y \in \mathcal{F}\}$.

The important survey on convex geometries is [5]. In particular, it defines a combinatorial parameter of *convex dimension*.

Also, an important example of convex geometry is described in the survey: a geometry of convex point configurations, also known as an *affine geometry*. In this example, X is a (finite) set of points in $\mathbb{R}^n$, and $Y \in \mathcal{F}$ iff convex hull generated by Y does not have points from $X \setminus Y$. Such subsets of X are called *convex*.

Example 1.

Let $X = \{a, b, c, d\}$ and $\mathcal{F} = \{\emptyset, a, b, c, d, ab, bc, cd, abc, bcd, X\}$. It is straightforward to check that $(X, \mathcal{F})$ is a convex geometry. Moreover, one can verify that if points a, b, c, d are placed on the line in alphabetical order, then $\mathcal{F}$ will be exactly the family of convex subsets of X, thus, $(X, \mathcal{F})$ is affine due to such point configuration representation.

Convex dimension of this geometry is 2, because there are two *chains* of convex sets: $\emptyset \subset a \subset ab \subset abc \subset X$ and $\emptyset \subset d \subset cd \subset bcd \subset X$ such that any convex subset is the intersection of subsets from these two chains.

It is known [6, 8] that every finite convex geometry is *embedded* into some affine one, but it may require high dimension of underlying Euclidean space $\mathbb{R}^n$.

To decrease the dimension of space needed for representation, one might consider a family of convex shapes X instead of just points. The family $\mathcal{F}$, again, follows the rule that $Y \in \mathcal{F}$ iff the convex hull formed by all points in all $y \in Y$ does not contain any $z \in X \setminus Y$. Not every family of segments on the plane will form a convex geometry that way, but every family of circles will.

In [4] it was shown that every (finite) convex geometry of convex dimension 2 is representable by circles on the plane. Initial hypothesis that it is possible to do for all (finite) convex geometries was disproved by a counterexample in [1], which had convex dimension 6.

This window for convex dimension was recently shortened in [7], where other examples of geometries of convex dimension 4 and 5 were found, for which representation by circles on the plane was not possible.

This leaves the question about convex dimension 3 open: is it possible to represent those geometries by circles on the plane? According to results of [7], for any counterexample one would need to look among geometries with $|X| > 5$.

There is a possibility that geometries of convex dimension 3 are actually representable by circles on the plane. We note that result of [4], while it was proved for circles, also works for segments on the line. In this case, maybe even the generalization will be possible for representation of geometries with convex dimension $n+1$ by the balls in $\mathbb{R}^n$, which would be the first nice connection between combinatorial parameter of convex dimension and the dimension of space for representation.

References

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