2022 Baruch Discrete Math REU: Sample Problem

Ballistic annihilation is a one-dimensional interacting particle system introduced by physicists in the 1980s as a caricature of more complex systems exhibiting anomalous compound-decay [6]. Particles are placed throughout the real line and assigned velocities at which they move from the onset. When two particles collide, they mutually annihilate and are removed from the system. While a variety of velocity distributions have been studied, what has become the canonical way to assign velocities is independently from $\{-1, 0, 1\}$ where velocity 0 is assigned with probability $p \in [0, 1)$, and velocities ± 1 symmetrically with probability $(1-p)/2$. We will refer to this system as **symmetric three-velocity ballistic annihilation** (BA).

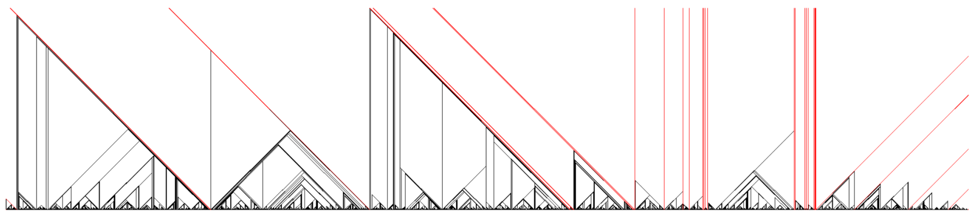

Figure 1. A graphical representation of symmetric three-velocity ballistic annihilation with $p = 1/4$. Time is the y-axis and space is the x-axis.

Many intriguing features of BA were inferred by physicists in the 1990s [4, 9]. Perhaps the most fundamental quantity is the critical initial density of velocity-0 particles below which no particles persist for all time:

$$p_c = \sup\{p \colon \text{no particles survive}\}.$$

It was conjectured that $p_c = 1/4$ for BA, but at the time was never resolved in a rigorous probabilistic manner. A major difficulty is that the order in which collisions occur is sensitive to perturbations; changing the velocity of a single particle can have a cascading effect. This makes it difficult to compare processes with different parameters. In 2018 there was a breakthrough from Haslegrave, Sidoravicius, and Tournier that rigorously proved $p_c = 1/4$ [7]. Seeking to better understand the reach, as well as limits, of the approach in [7], Junge and various coauthors have been working to extend this result to more general systems [8, 2, 5].

Inspired by earlier work from physicists [3], Benitez, Junge, Lyu, Redman, and Reeves extended BA dynamics to include reactions that produce new particles [1]. Denote left-moving, right-moving and stationary particles by $\overleftarrow{\bullet}, \overrightarrow{\bullet}$, and $\dot{\bullet}$, respectively. Fix parameters $0 \le a, b, x < 1$ with $a + b \le 1$. Using the notation $[\bullet - \bullet \implies \Theta, \quad \theta]$ to denote a collision resulting in an outcome $\Theta \in \{\dot{\bullet}, \overrightarrow{\bullet}, \overleftarrow{\bullet}, \varnothing\}$ with probability θ, we define **three parameter coalescing ballistic annihilation** (TCBA) as the family of systems with collision rules:

$$\overrightarrow{\bullet} - \overleftarrow{\bullet} \implies \begin{cases} \overleftarrow{\bullet}, & a/2 \\ \overrightarrow{\bullet}, & a/2 \\ \dot{\bullet}, & b \\ \varnothing, & 1-(a+b) \end{cases} \qquad \dot{\bullet} - \overleftarrow{\bullet} \implies \begin{cases} \overleftarrow{\bullet}, & x \\ \varnothing, & 1-x \end{cases} \qquad \overrightarrow{\bullet} - \dot{\bullet} \implies \begin{cases} \overrightarrow{\bullet}, & x \\ \varnothing, & 1-x \end{cases}.$$

See Figure 2 for an example. Note that BA is the special case $a = b = x = 0$.

The main result in [1] was a general formula for the analogue of p_c for TCBA. The formula is not obviously inferred from the model definition and underscores the complexity of these coalescing systems.

FIGURE 2. Graphical representations of TCBA (left) versus BA (right). Despite the same initial configuration, markedly different outcomes can occur.

Theorem 1. *For any TCBA it holds that*

$$p_c(a,b,x) = \frac{1-b(1-x)}{4-3x-(a+b)(1-x)}.$$

A further direction is to study coalescing ballistic annihilation with the additional coalescing reaction:

$$[\dot{\bullet} - \vec{\bullet} \Longrightarrow \dot{\bullet}, \quad y] \quad \text{and} \quad [\vec{\bullet} - \dot{\bullet} \Longrightarrow \dot{\bullet}, \quad y] \tag{1}$$

for some parameter $y \in [0,1]$ with $0 \leq x + y \leq 1$. In words, stationary particles survive collisions with probability y. This reaction was not included in [1], because it induces an otherwise not-present asymmetry in the analysis. The specifics are more complicated, but the basic idea is that one must distinguish between visits to the origin by moving particles that would destroy a blockade there, versus those that would not. Call such visits strong and weak, respectively At a key point in the argument [1] it is important to compute the probability that the first arrival to the origin by a right-moving particle occurs before the first left-moving particle to arrive. In TCBA, this has probability 1/2 by symmetry. However, the reaction (1) makes it necessary to compare the first strong visit and the first weak visit, which does not enjoy an obvious symmetry. An interesting direction is to attempt to analyze this case, which appears to go beyond the "exactly solvable" setting of [7] and [1].

Question 1. *Estimate $p_c(a,b,x,y)$ in systems that include the reactions at* (1).

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