Project topics:
Discrete geometry and topological combinatorics

Over the course of several years, the REU projects in which I have been the mentor have one common topic: **studying which combinatorial problems can be tackled with non-combinatorial methods,** In particular, I am interested in the applications of topology and linear algebra in combinatorics.

It is not clear why the solutions to some combinatorial problems benefit from borrowing methods in other areas. Trying to figure this out can mean different things. In some cases, this means finding new problems that can be solved this way, and in other cases we can simplify existing proofs to avoid doing this.

Discrete geometry is an area in which topological and linear-algebraic methods arise naturally. Understanding the combinatorial properties of geometric sets is an interesting problem. The sets can be finite sets of points, affine flats, or even general convex sets (a set is convex if it contains the segment between any two of its points).

Instead of giving example projects for this summer, I will describe below the three main topics I use for REU projects. You can find a good list of projects from past years here, which should give you a good sense of what we might work on during the summer. If you are curious about these topics, I also have a free set of notes here. The precise topics for the projects will be shared later with potential REU participants. The main three topics in which I lead projects are the following:

- **Projects related to the intersection patterns of convex sets**. A classic theorem in the area is Helly's theorem.

Theorem 1 (Helly 1921)**.** *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d+1$ or fewer sets in $\mathcal{F}$ intersect, then all of $\mathcal{F}$ intersects.*

Many successful projects have focused on finding variations and generalizations of Helly's theorem (colorful, fractional, quantitative, etc...).

- **Projects related to the combinatorial properties of finite families of points in $\mathbb{R}^d$.** Here is one of my favorite theorems:

Theorem 2 (Tverberg 1966)**.** *Given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of them into r sets whose convex hulls intersect.*

The convex hull of a family of points is the smallest convex set that contains it. One interesting feature of Tverberg's theorems is that both linear-algebraic and topological tools have been used to prove it, along with different generalizations. This result is really at the crossroad of linear algebra and topology, so studying its many generalizations helps us understand these interactions.

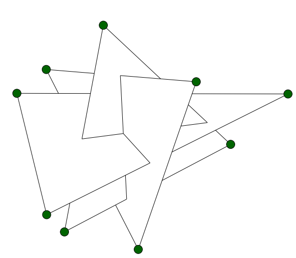

Figure 1: Example of a Tverberg partition with $d=3, r=3$. For any 9 points in $\mathbb{R}^3$ we can find a partition as above. Note that it might not be three flat triangles every time.

• **Projects related to fair partitions.** Again, let us start with the quintessential theorem in the area.

Theorem 3 (The "ham sandwich theorem", Banach 1938)**.** *Suppose we are given d finite sets of points in $\mathbb{R}^d$, each with an even number of points and so that no $d+1$ of the given points lie in a hyperplane. Then, there exists a hyperplane that leaves exactly half of each of the d sets on either side.*

There is a large number of variations of this theorem. We can change the number of sets we want to divide $\mathbb{R}^d$ into, the geometric rules for the partition, or our notion of "fairness" (maybe we don't require each set to be halved exactly).

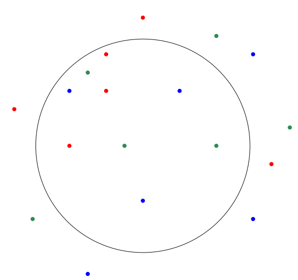

Figure 2: A neat consequence of the ham sandwich theorem. For any three sets of points in $\mathbb{R}^2$, there exists a circle that contains exactly half of each set.

These kind of partitioning results are known for their topological solutions (the ham sandwich theorem, for example, uses the Borsuk–Ulam theorem). Projects in this area aim to determine the validity of new mass partitioning results.