EXAMPLE PROJECT FOR 2023 REU

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In 2023 I will have projects on combinatorial geometry. This is a subject in which we study how geometric properties of sets can affect geometry. For example, we can study the intersection patterns of axis-parallel boxes or convex sets in general. How would you define that two sets of 10 points on the plane are "combinatorially distinct"? How many pairwise combinatorially distinct sets of 10 points on the plane are there?

The projects this year will revolve around Helly's theorem, Tverberg's theorem, and the Ham Sandwich theorem. Below is a short explanation of some problems around Helly's theorem. The ham sandwich theorem is a result about fair divisions. In the space, it implies that *for any* 3 *sets of points in* $\mathbb{R}^3$*, each with an even number of points, all in general position, there is a plane that splits each of them exactly by half.*

The ham sandwich theorem is very close to **topological combinatorics**, which is an area that studies how topological techniques can be used to solve problems in combinatorics. Helly's theorem and Tverberg's theorem also have extensions whose solutions involve deep topological techniques. The projects I have for Helly and Tverberg this year are closer to their linear versions.

If you are interested in the general topics, I suggest skimming the second chapter of the notes mentioned on the REU site and stopping to read the details of any theorem that catches your attention. Here is an example of some research directions around Helly's theorem.

PROJECT INTRODUCTION: Discrete Helly and Piercing numbers

Helly's theorem says that *a finite family of convex sets in* $\mathbb{R}^d$ *has a point of intersection if and only if every subfamily of at most* $d+1$ *sets has a point of intersection.* In other words, in order to check for an intersection of the whole family it is sufficient to check the intersection of $(d+1)$-tuples of sets. There are many variations of this result, and its connection to optimization algorithms is important in computer science.

Helly's theorem guarantees that the intersection of a family of convex sets will be non-empty, but if we are given a point in each of the intersecting $d+1$-tuples, it is still possible that none of those are in the intersection of the whole family.

If we impose further restrictions on the sets, it might be possible to avoid that problem.

Problem. *Let* $\mathcal{F}$ *be a finite family of half-spaces in* $\mathbb{R}^d$ *and* P *be a finite set of points in* $\mathbb{R}^d$*. If the intersection of every* $d+1$ *sets in* $\mathcal{F}$ *contains a point in* P*, can we guarantee that we can pierce every set of* $\mathcal{F}$ *with at most d points of* P*?*

The case $d = 2$ of the result above has recently been proved affirmatively. Similar results also holds for other shapes, such as disks in the plane or axis-parallel boxes in $\mathbb{R}^d$. There are plenty of variations of Helly's theorem (colorful versions, fractional versions, etc.). These variations seem to be mostly unexplored for these "discrete" versions of Helly's theorem, so there are many directions to go to in this project.

Given a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, finding the piercing number of $\mathcal{F}$, the minimum number of points needed to intersect every set in $\mathcal{F}$, is an important parameter in combinatorial geometry. The discrete versions of Helly's theorem allow us to take a new look at this parameter and find situations in which improved bounds can be found.

A key extension of Helly's theorem is the *colorful* version of Lovász, that says the following.

Theorem 1. *Let* $F_1, \dots, F_{d+1}$ *be* $d+1$ *finite families of convex sets in* $\mathbb{R}^d$*. If each time we pick one set of each family we make a* $(d+1)$*-tuples that has a non-empty intersection, then at least one* F_i *has a non-empty intersection.*

This is called the colorful version because we can think of each F_i as a family of sets of a particular color. Then, if every colorful family has a non-empty intersection, some monochromatic family must have non-empty intersection. It implies the classic version by taking $F_1 = \cdots = F_{d+1}$.

In this project, we will also explore colorful discrete Helly theorems for different families of sets.

Warm-up problems

Problem. *Show that the problem above has a negative answer for general convex sets. In other words, additional geometric restrictions are necessary.*

Problem (Helly for boxes)**.** *Let* $\mathcal{C}$ *be a finite family of axis-parallel boxes in* $\mathbb{R}^d$*. Prove that if any two such boxes have a point in common, then all of them do.*

Surveys/resources for Helly's theorem : [HW17, ALS17]

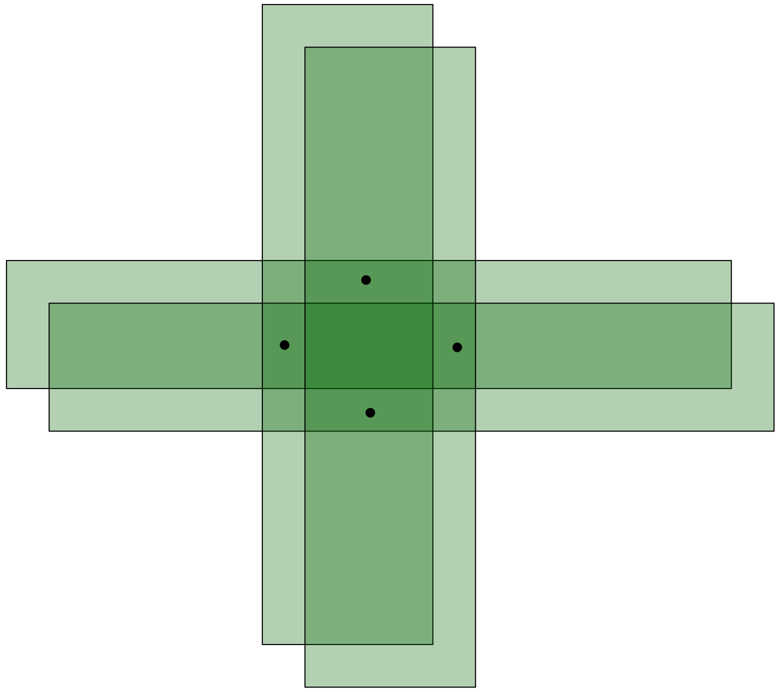

Figure 1: The intersection of every three boxes contains one of the marked points, but the intersection of the whole family does not.

References

[ALS17] Nina Amenta, Jess A. De Loera, and Pablo Sobern, *Hellys theorem: New variations and applications*, American Mathematical Society, vol. 685, American Mathematical Society, 2017.

[HW17] Andreas F. Holmsen and Rephael Wenger, *HELLY-TYPE THEOREMS AND GEOMETRIC TRANSVERSALS*, Handbook of Discrete and Computational Geometry, Chapman and Hall/CRC, 3rd ed., 2017, pp. 91–123.