EXAMPLE PROJECTS FOR 2022 REU

In 2022 I will have projects related to Tverberg's theorem and to Helly's theorem. Below I describe a Helly-type project. If you are interested in learning about Tverberg's theorem and its connections to combinatorics, discrete geometry, and topological combinatorics, see [BS18].

For results from previous REU participants around Helly's and Tverberg's theorems see, for example, those of Sherry Sarkar, Alexander Xue, Travis Dillon, Yaqian Tang, Jack Messina, of João Pedro Carvalho.

PROJECT OUTLINE: Discrete Helly theorems revisited.

Helly's theorem is one of the first results about combinatorial properties of convex sets. It says that *a finite family of convex sets in* $\mathbb{R}^d$ *has a point of intersection if and only if every subfamily of at most* $d+1$ *sets has a point of intersection.* In other words, in order to check for an intersection of the whole family it is sufficient to check the intersection of $(d+1)$-tuples of sets. There are many variations of this result, and its connection to optimization algorithms is important in computer science.

Helly's theorem guarantees that the intersection of a family of convex sets will be non-empty, but if we are given a point in each of the intersecting $d+1$-tuples, it is still possible that none of those are in the intersection of the whole family.

If we impose further restrictions on the sets, it might be possible to avoid that problem.

Problem. *Let* $\mathcal{F}$ *be a finite family of half-spaces in* $\mathbb{R}^d$ *and* P *be a finite set of points in* $\mathbb{R}^d$*. If the intersection of every* $d+1$ *sets in* $\mathcal{F}$ *contains a point in* P*, can we guarantee that we can pierce every set of* $\mathcal{F}$ *with at most* d *points of* P*?*

The case $d = 2$ of the result above has recently been proved affirmatively. Similar results also holds for other shapes, such as disks in the plane or axis-parallel boxes in $\mathbb{R}^d$. There are plenty of variations of Helly's theorem (colorful versions, fractional versions, etc.). These variations seem to be mostly unexplored for these "discrete" versions of Helly's theorem, so there are many directions to go to in this project.

Warm-up problems

Problem. *Show that the problem above has a negative answer for general convex sets. In other words, additional geometric restrictions are necessary.*

Problem (Helly for boxes)**.** *Let* $\mathcal{C}$ *be a finite family of axis-parallel boxes in* $\mathbb{R}^d$*. Prove that if any two such boxes have a point in common, then all of them do.*

Surveys/resources for Helly's theorem : [HW17, ADLS17]

References

[ADLS17] Nina Amenta, Jesús A. De Loera, and Pablo Soberón, *Helly's theorem: New variations and applications*, Algebraic and geometric methods in discrete mathematics, American Mathematical Society, Providence, Rhode Island, 2017, pp. 55–95.

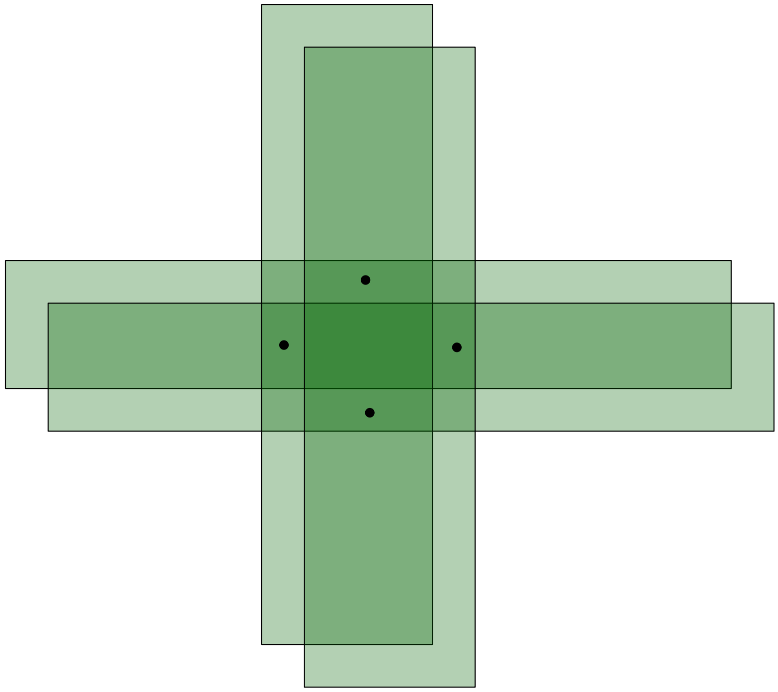

Figure 1: The intersection of every three boxes contains one of the marked points, but the intersection of the whole family does not.

[BS18] Imre Bárány and Pablo Soberón, *Tverberg's theorem is 50 years old: A survey*, Bulletin of the American Mathematical Society **55** (2018), no. 4, 459–492.

[HW17] Andreas F. Holmsen and Rephael Wenger, *HELLY-TYPE THEOREMS AND GEOMETRIC TRANSVERSALS*, Handbook of Discrete and Computational Geometry, books.google.com, 2017, pp. 91–123.