Project Example: Point–Line Incidences with Restrictions

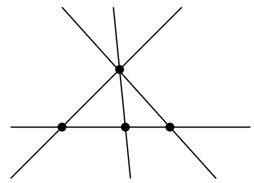

Figure 1: A configuration of four points, four lines, and nine incidences.

Given a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines, both in $\mathbb{R}^2$, an *incidence* is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ such that the point p is contained in the line ℓ. We denote by $I(\mathcal{P}, \mathcal{L})$ the number of incidences in $\mathcal{P} \times \mathcal{L}$. For example, Figure 1 depicts a configuration with nine incidences. For any m and n, Erdős constructed a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines with $\Theta(m^{2/3}n^{2/3} + m + n)$ incidences. In 1983, Szemerédi and Trotter [6] proved that this number of incidences is asymptotically maximal.

Theorem 1 (The Szemerédi-Trotter theorem)**.** *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

There are many variants of this problem. For example, one may ask for the maximum number of incidences between points and circles, or parabolas, incidences in higher dimensions, in spaces over other fields, and so on. Almost all of these problems are still wide-open.

One reason for studying so many different incidence problems is that they are surprisingly useful. Many problems in discrete geometry can be reduced to incidence problems. Surprisingly, this is also the case for problems in additive combinatorics, harmonic analysis, theoretical computer science, number theory, and more (for lists of such applications, see [3, 4]).

What configurations of n points and n lines lead to $\Theta(n^{4/3})$ incidences? Here is one of the only two examples we have, by Elekes [1].

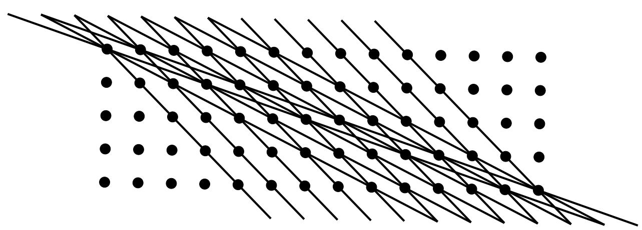

Figure 2: Elekes's construction.

Elekes's construction. Let $r = (n/4)^{1/3}$ and $s = (2n)^{1/3}$ (for simplicity, instead of taking the ceiling function of s and r, we assume that these are integers). We set

$$\mathcal{P} = \{ \ (i, j) \ : \ 1 \leq i \leq r \quad \text{and} \quad 1 \leq j \leq 2rs \ \},$$

and

$$\mathcal{L} = \{ \ y = ax + b \ : \ 1 \leq a \leq s \quad \text{and} \quad 1 \leq b \leq rs \ \}.$$

Note that this construction consists of a rectangular section of the integer lattice and of a "lattice" of lines; such a configuration is depicted in the above figure, rotated by 90°. We have that

$$|\mathcal{P}| = 2r^2 s = 2 \cdot \frac{n^{4/3}}{(4n)^{2/3}} \cdot \frac{(2n^2)^{1/3}}{n^{1/3}} = n,$$

and

$$|\mathcal{L}| = rs^2 = \frac{n^{2/3}}{(4n)^{1/3}} \cdot \frac{(2n^2)^{2/3}}{n^{2/3}} = n.$$

Consider a line $\ell \in \mathcal{L}$ that is defined by the equation $y = ax + b$, for some valid values of a and b. For any $x \in \{1, \ldots, r\}$, there exists a unique $y \in \{1, \ldots, 2rs\}$ such that the point (x, y) is incident to ℓ. That is, every line of $\mathcal{L}$ is incident to exactly r points of $\mathcal{P}$, so

$$I(\mathcal{P}, \mathcal{L}) = r \cdot |\mathcal{L}| = \frac{n^{2/3}}{(4n)^{1/3}} \cdot n = 2^{-2/3} n^{4/3}.$$

□

Let us return to our question: What configurations of n points and n lines lead to $\Theta(n^{4/3})$ incidences? Must the point set be a lattice? Can we have a lattice of size different than $n^{1/3} \times n^{2/3}$? Must there exist a line that contains $\Omega(n^{1/2})$ of the points? While Theorem 1 is known from the early 1980's, not much is known about this structural problem.

The only non-trivial piece of information known is the following result of Solymosi [5].

Theorem 2. *For every constant integer k, the following holds for every sufficiently large n. Let $\mathcal{P}$ be a set of n points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$, such that there are $\Theta(n^{4/3})$ incidences in $\mathcal{P} \times \mathcal{L}$. Then exists a subset $\mathcal{P}' \subset \mathcal{P}$ of k points, no three on a line, such that there is a line of $\mathcal{L}$ passing through each of the $\binom{k}{2}$ pairs of $\mathcal{P}' \times \mathcal{P}'$.*

Intuitively, Theorem 2 says: If the point–line configuration does not contain some constant-sized sub-configuration, then the number of incidences is[1] $o(n^{4/3})$.

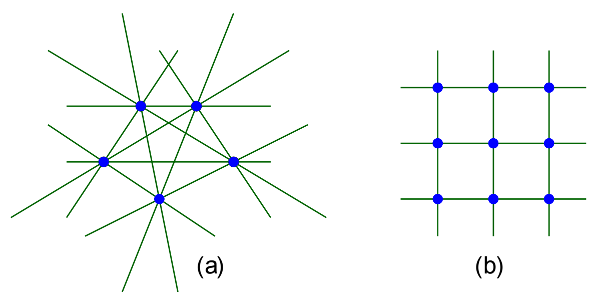

Figure 3: (a) The configuration forbidden in Theorem 2. (b) The configuration forbidden in Theorem 3.

Recently, Mirzaei and Suk [2] derived a stronger incidence bound for a more special case. For a positive integer t, denote by X_t the following point-line configuration: Two sets $\mathcal{L}_1, \mathcal{L}_2$, each of t lines, such that no line exists in both sets, no three lines intersect at a point, and each of the t^2 intersection points of $\mathcal{L}_1 \times \mathcal{L}_2$ is in the point set. See Figure 3(b) for an example.

[1]The $o(\cdot)$-notation means asymptotically smaller than the expression in the parentheses.

Theorem 3. *Let $\mathcal{P}$ be a set of n points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. For any positive integer t, if the two sets do not have X_t as a subconfiguration, then*

$$I(\mathcal{P}, \mathcal{L}) = O\left(n^{\frac{4}{3}-\frac{1}{9t-6}}\right).$$

In the other direction, Mirzaei and Suk prove that there exists a configuration with no X_2 and $\Omega(n^{15/14})$ incidences.

The project. This project involves further studying the topic of point–line incidences with restrictions. It involves looking for other cases where one can improve the bound of Theorem 2. In the other direction, it involves find constructions that have many incidences while not containing some sub-configurations. Adam has some specific approaches in mind, but prefers not to make those available publicly.

References

[1] G. Elekes, On the number of sums and products, *ACTA ARITHMETICA — WARSZAWA* **81** (1997), 365–367.

[2] M. Mirzaei, and A. Suk, On grids in point-line arrangements in the plane, arXiv:1812.11162.

[3] A. Sheffer, Incidences Outside of Discrete Geometry, `https://adamsheffer.wordpress.com/2016/01/22/incidences-outside-of-discrete-geometry/`

[4] A. Sheffer, Incidences Outside of Discrete Geometry, part 2, `https://adamsheffer.wordpress.com/2017/10/06/incidences-outside-of-discrete-geometry-part-2/`

[5] J. Solymosi, Dense arrangements are locally very dense I, *SIAM Journal on Discrete Mathematics* **20** (2006), 623–627.

[6] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.